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HKR theorem for smooth S-algebras

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Abstract

We derive an étale descent formula for topological Hochschild homology and prove a HKR theorem for smooth S-algebras.

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1. Introduction

One of the main results for computing the Hochschild homology of smooth discrete algebras is the Hochschild–Kostant–Rosenberg (HKR) theorem (e.g. see Chapter 3 of [8]), which states that for a smooth algebra $k \to A$, the Hochschild homology coincides with differential forms:

$$HH_*(A) = \Omega^*_{A|k}.$$

In fact this result is often used not only to compute the Hochschild homology, but also the other way around: in order to generalize some results to non-smooth (or even non-commutative) algebras one replaces the differential forms by Hochschild homology. Other applications include a comparison theorem between cyclic and de Rham homology theories.

One of our objectives is to develop a topological analogue of the HKR theorem in the framework provided in [5], or more precisely, in the category of commutative of S-algebras. Recall that S-algebras are equivalent to the more traditional notion of

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E_∞-ring spectra, and are a generalization to stable homotopy theory of the algebraic notion of a commutative ring. In this context, the topological André–Quillen homology of a commutative S-algebra A is the natural replacement of the module of differentials $\Omega^1_{A|k}$, as it is evident from the definition of TAQ. The definitions of TAQ, as well as THH, in our context are recalled in Section 2, and we refer to [1] and Chapter IX of [5] for detailed discussion of these notions. Noting that the orbits of the n'th smash powers of the suspension of TAQ, $(\Sigma TAQ(A))^{\wedge A^n}/\Sigma_n$, are analogous to symmetric powers in the graded context, and therefore correspond to taking exterior powers (and thus are the analogues of the higher-order modules of differentials), we state our main theorem.

Theorem 1.1. (HKR). *For a connective smooth S-algebra A, the natural (derivative) map $THH(A) \to \Sigma TAQ(A)$ has a section in the category of A-modules which induces an equivalence of A-algebras*:

$$\mathbb{P}_A \Sigma TAQ(A) \xrightarrow{\simeq} THH(A),$$

where $\mathbb{P}$ is the symmetric algebra triple.

The following is a description of the structure of the paper.

In Section 2 we recall the definitions of topological Hochschild homology and topological André–Quillen homology in our framework. More precisely, the two main categories where our work takes place are the following. The first one is the category of A-modules, denoted by $\mathscr{M}_A$, where A is a commutative S-algebra. There is a triple $\mathbb{P}_A : \mathscr{M}_A \to \mathscr{M}_A$ on this category given by $\mathbb{P}M = \bigvee_{j \geqslant 0} M^j/\Sigma_j$ (here M^j denotes the j-fold smash power over A and $M^0 = A$), which leads us to the second category of interest—the category $\mathscr{M}_A[\mathbb{P}]$ of algebras in $\mathscr{M}_A$ over $\mathbb{P}$. Clearly, it is equivalent to the category of commutative A-algebras $\mathscr{C}_A$. For convenience, we denote the reduced version of $\mathbb{P}$ by $\mathbb{P}^1$. In other words, $\mathbb{P}^1$ is the obvious functor for which $\mathbb{P} = A \vee \mathbb{P}^1$.

Note that both of these categories are closed model categories, and for a discussion on their homotopy categories we refer to Chapter VII of [5]. A good account for the general theory of closed model structures can be found in [3].

In Sections 3 and 4 we define étale, thh-étale, smooth and thh-smooth S-algebras, show that all these are generalizations of appropriate notions from discrete algebra, and prove their basic properties.

Section 5 is devoted to establishing some conditions on a simplicial set X_* and a map of commutative R-algebras $A \to B$ that imply the identity

$$A \otimes X_* \wedge_A B \simeq B \otimes X_*. \tag{1}$$

Observe that as a special case of this equation (more precisely, when we take the simplicial set X_* to be the circle S^1_*), we get an equation

$$THH(A) \wedge_A B \simeq THH(B). \tag{2}$$

Here we employed the identity $THH(A) \simeq A \otimes S^1_*$ derived by McClure et al. in [12]. Of course, in discrete algebra, the analogue of (2) is referred to as *étale descent formula for HH* (see e.g. [6]). Following this, we will refer to both (1) and (2) as étale descent formulas.

To prove (1), we produce a necessary condition for it to hold, and show that under some additional hypothesis, that condition is also sufficient. The notion of completeness is also discussed here, as it plays an important role in understanding (1). Eq. (2) is a key technical step in the proof of the HKR theorem for smooth S-algebras.

In Section 6, we prove the main (HKR) Theorem 1.1, and conclude the section by showing that, as a consequence of the HKR theorem, the first fundamental sequence of the modules of differentials splits under a smoothness hypothesis. Here, following the terminology of discrete algebra, by first fundamental sequence of the modules of differentials we mean the homotopy cofibration sequence

$$TAQ^R(A) \wedge_A B \to TAQ^R(B) \to TAQ^A(B),$$

associated to the sequence $R \to A \to B$ of S-algebras (see [1] for a detailed discussion on this).

Our definition for a map of commutative ring spectra $f: C \to D$ being étale when $TAQ(D|C) \simeq *$ is not new. We were first introduced to this idea by F. Waldhausen in 1991. Some other people whom we are aware of using this idea (either formally or in private conversation) are: M. Basterra, T. Goodwillie, T. Hunter, J. Klein, I. Kriz, M. Mandell, J. McClure, T. Pirashvili, C. Rezk, B. Richter, A. Robinson, J. Rognes, J. Smith, and S. Whitehouse. The idea of thh-étale that we use seems fairly common to the extent that most of these people have considered this also. In particular, recent work by J. Rognes independently establishes several of the structural properties of thh-étale maps which we use.

We have been greatly aided by many mathematicians while working out our ideas for this paper. In particular, we would like to thank Maria Basterra for teaching us about commutative S-algebras and how to work with them. We thank Mike Mandell for his support, insights and important examples. This work arose from a series of talks with Charles Rezk (who also caught a serious mistake in an earlier draft) while he taught us about the DeRham cohomology of commutative ring spectra. We came upon the main conjecture while talking with Birgit Richter and were certainly motivated by ideas of Nick Kuhn about splitting Goodwillie Taylor towers.

2. Preliminaries: THH and TAQ of commutative S-algebras

In this section we give a brief introduction into THH and TAQ of commutative S-algebras. Chapter IX of [5,1] provide a good in depth discussion of these notions in our framework.

Let R be a cofibrant commutative S-algebra, A—a cofibrant R-algebra or a cofibrant commutative R-algebra, and M an (A,A)-bimodule. Write A^p for the p-fold $\wedge_R$-power, and let

$$\phi: A \wedge_R A \to A \quad \text{and} \quad \eta: R \to A$$

be the product and unit of A-respectively.

Let

$$\xi_l: A \wedge_R M \to M \quad \text{and} \quad \xi_r: M \wedge_R A \to M$$

be the left and right actions of A on M. Denote the canonical cyclic permutation isomorphism by τ:

$$\tau : M \wedge_R A^p \wedge_R A \to A \wedge_R M \wedge_R A^p.$$

Definition 2.1. Let $THH^R(A;M)_*$ be the simplicial R-module whose R-module of p-simplices is $M \wedge_R A^p$, and whose face and degeneracy operators are

$$d_i = \begin{cases} \xi_r \wedge (id)^{p-1} & \text{if } i = 0, \\ id \wedge (id)^{i-1} \wedge \phi \wedge (id)^{p-i-1} & \text{if } 1 \leqslant i < p, \\ (\xi_l \wedge (id)^{p-1}) \circ \tau & \text{if } i = p, \end{cases}$$

$$s_i = id \wedge (id)^i \wedge \eta \wedge (id)^{p-i}.$$

Define

$$THH^R(A;M) = |THH^R(A;M)_*|.$$

When $M = A$, we delete it from the notation, writing $THH^R(A)$.

Clearly this definition [5] mimics the definition of the standard complex for the computation of Hochschild homology, as given in [2]. Of course, the passage from a simplicial spectrum to its geometric realization is the topological analogue of passage from a simplicial k-module to a chain complex.

Observe that the maps

$$\xi_p = id \wedge \eta^p : M \simeq M \wedge_R R^p \to M \wedge_R A^p$$

induce a natural map of R-modules

$$\xi = |\xi_*| : M \to THH^R(A;M).$$

If A is a commutative R-algebra, then clearly $THH^R(A)_*$ is a simplicial commutative R-algebra and $THH^R(A;M)_*$ is a simplicial $THH^R(A)$-module. Hence, $THH^R(A)$ is a commutative A-algebra with the unit map given by the above map $\xi : A \to THH^R(A)$.

Observe that if M is an (A,A)-bimodule and $\underline{M}$ is the corresponding constant simplicial (A,A)-bimodule, then

$$M \wedge_{A^e} \beta^R(A) \cong |\underline{M} \wedge_{A^e} \beta^R_*(A)|,$$

where $A^e = A \wedge A^{op}$. We have canonical isomorphisms

$$M \wedge_R A^p \cong M \wedge_{A^e} (A^e \wedge_R A^p) \cong M \wedge_{A^e} (A \wedge_R A^p \wedge_R A)$$

given by permuting $A^{op} = A$ past A^p. As these isomorphism commute with the face and degeneracy operations, we get

$$THH^R(A;M) \cong M \wedge_{A^e} \beta^R(A). \tag{3}$$

Now we turn our attention to the Topological André–Quillen Homology. The definition, presented by Maria Basterra in [1], employs the following two functors.

The augmentation ideal functor: Let A be a commutative S-algebra, and $I : \mathscr{C}_{A/A} \to \mathscr{N}_A$ the functor from the category of commutative A-algebras over A to the category of A-NUCA's which assigns to each algebra $(B, \eta : A \to B, \varepsilon : B \to A)$ its "augmentation ideal": $I(B)$ defined by the pullback diagram in $\mathcal{M}_A$,

$$\begin{tikzcd} I(B) \arrow[r] \arrow[d] & B \arrow[d, "\varepsilon"] \\ * \arrow[r] & A. \end{tikzcd}$$

Note that by the universal property of pullbacks $I(B)$ comes with a commutative associative (not necessarily unital) multiplication. Moreover, this functor has a left adjoint $K : \mathcal{N}_A \to \mathcal{C}_{A/A}$ which maps a non-unital algebra N to $N \vee A$ (Proposition 3.1 of [1]). In addition this adjunction produces an equivalence of homotopy categories given by the total derived functors $\mathbf{L}K$ and $\mathbf{R}I$ (Proposition 3.2 of [1]).

The indecomposables functor: Let $Q : \mathcal{N}_A \to \mathcal{M}_A$ denote the "indecomposables" functor that assigns to each N in $\mathcal{N}_A$ the A-module $Q(N)$ given by the pushout diagram in $\mathcal{M}_A$

$$\begin{tikzcd} N \wedge_A N \arrow[r] \arrow[d] & * \arrow[d] \\ N \arrow[r] & Q(N). \end{tikzcd}$$

This functor has a right adjoint $Z : \mathcal{M}_A \to \mathcal{N}_A$ given by considering A-modules as non-unital algebras with zero multiplication. Since Z is the identity on morphisms and the closed model structure on $\mathcal{N}_A$ is created in $\mathcal{M}_A$, Z preserves fibrations and acyclic fibrations, so by Chapter 9 of [3], the total derived functors $\mathbf{R}Z$ and $\mathbf{L}Q$ exist and are adjoint.

Definition 2.2. Let $B \to A$ be a map of commutative S-algebras. Define

$$TAQ(B/A) = \Omega_{B/A} \stackrel{\text{def}}{=} \mathbf{L}Q\mathbf{R}I(B \wedge_A^{\mathbf{L}} B),$$

where $B \wedge_A^{\mathbf{L}} B$ denotes the total derived functor of $- \wedge_A^{\mathbf{L}} B$ evaluated at B.

Of course, as it is observed in [1], $\Omega_{B/A}$ is simply a derived analogue of the B-module of Kähler differentials from classical algebra.

Notation. Fix a cofibrant commutative S-algebra A. Then for an A-algebra B and an A-module M, we denote by $THH(B, M|A)$ and $TAQ(B, M|A)$ the topological Hochschild and André–Quillen homologies of B over A with coefficients in M. If $M = B$, we omit it from the notation. In addition, $\widetilde{\mathrm{THH}}(B|A)$ stands for the reduced topological Hochschild homology, defined to be the homotopy cofiber of the natural map $B \to THH(B|A)$.

3. (thh-)étale S-algebras

Recall that in discrete algebra smooth maps can be roughly defined to be the maps which can be decomposed into a polynomial extension followed by an étale extension.

Definition 3.1. We say that a discrete k-algebra A is smooth if for any prime ideal of A there is an element f not in that prime such that there exists a factorization

$$k \to k[x_1,\ldots,x_n] \xrightarrow{\phi} A_f$$

with ϕ étale, i.e. flat and unramified.

Under some finiteness and flatness conditions this notion of smooth maps coincides with most other standard ones (see the appendix of [8]). It is with this approach to smoothness in mind that we define our smooth maps of S-algebras. Hence the need to discuss the notion of étale algebras first. Recall that for discrete algebras, both smooth and étale maps are defined to be finite in some appropriate sense. We do not impose a finiteness condition on S-algebras as it is not needed for our main results. Consequently, a more appropriate terminology to use would be 'formally' étale and smooth, which we do not for the sake of economy.

We begin with a pair of definitions. Let R be a commutative cell S-algebra and A, C and D commutative R-algebras.

Definition 3.2. The map of algebras $C \to D$ is étale (thh-étale) if $TAQ(D|C)$ is contractible ($D \xrightarrow{\simeq} THH(D|C)$).

We also define (thh-)étale coverings to be faithfully flat families of (thh-)étale extensions:

Definition 3.3. We say that $\{A \to A_\alpha\}_{\alpha\in\mathscr{I}}$ is a (thh-)étale covering of A if

1. each map $A \to A_\alpha$ is (thh-)étale, and
2. for each pair of A-modules $M \to N$ such that $M \wedge A_\alpha \to N \wedge A_\alpha$ is a weak equivalence for all α, the map $M \to N$ is itself a weak equivalence.

This definition gives rise to a few natural questions. Are there 'enough' (thh-)étale coverings? What is the relationship between étale and thh-étale?

Remark 3.4. We claim that for each commutative R-algebra A, at least one (non-trivial) étale covering and one (non-trivial) thh-étale covering exists. To see this, first recall some facts about localizing S-algebras.

Suppose T is a multiplicatively closed subset of $\pi_*(A)$. Then by Section 1 of Chapter V of [5], for each A-module M one can define a localization $M[T^{-1}]$ of M at T using a telescope construction with a key property

$$\pi_*(M[T^{-1}]) \cong \pi_*(M)[T^{-1}]. \tag{4}$$

Moreover, the localization of M is the smash product of M with the localization of A. In addition, by Theorem VIII. 2.1 of [5] one can construct the localization in such a way that $A[T^{-1}]$ is a cell R-algebra and the localization map $A \to A[T^{-1}]$ is an inclusion of a subcomplex. Moreover, since $A[T^{-1}]$ smashed over A with itself is equivalent to localizing $A[T^{-1}]$ at T, we conclude that $A[T^{-1}] \wedge_A A[T^{-1}] \cong A[T^{-1}]$, and hence the map $A \to A[T^{-1}]$ is (thh)-étale. Now for each prime of $\pi_*(A)$, pick an element f outside of it, and let T be the multiplicative system generated by that element. Let $M \to N$ be a map of A-modules such that $M_f \to N_f$ is an equivalence for all f. In other words, the induced map $\pi_*(M) \to \pi_*(N)$ is such that the localizations of this map are isomorphisms. Hence the map itself is an isomorphism (e.g. see Chapter 2 of [4]), proving that $\{A \to A_f\}$ is a covering. Of course, there are other collections of multiplicative systems in $\pi_*(A)$ that we can use to produce a covering (e.g. all the maximal ideals of $\pi_*(A)$); the key property is that if a map of modules localized at these systems is an isomorphism then the map itself is an isomorphism.

Recall that the Goodwillie derivative of THH is the suspension of TAQ and thus thh-étale implies étale. This is discussed in detail for example in [13]. While the converse is false in general, it does hold for certain classes of spectra; for example, the two notions are equivalent for connective spectra (see [13]). The following example (communicated by Mandell [10]) illustrates that étale does not always imply thh-étale.

Example 3.5. We work over the field $\mathbf{F}_p$. Fix $n > 1$ and let $C^*(K(\mathbf{Z}/p\mathbf{Z}, n))$ be the cochain complex of $K(\mathbf{Z}/p\mathbf{Z}, n)$ viewed as an E_∞-algebra. To ease the notation we denote this E_∞-algebra by R. R has a non-zero homotopy group in degree $-n$, while its $-n+1$st homotopy group is trivial. Recall that $THH(R|\mathbf{F}_p)$ is equivalent to $Tor^{R\otimes R}(R, R)$, hence we have an Eilenberg–Moore type spectral sequence (see Theorem IV.6.2 or Theorem IX.1.9 of [5]):

$$Tor_{p,q}^{\pi_*(R\otimes R)}(\pi_*(R), \pi_*(R)) \;\Rightarrow\; Tor_{p+q}^{R\otimes R}(R, R) = THH_{p+q}(R|\mathbf{F}_p).$$

Consequently, the $-n+1$st homotopy group of $THH(R|\mathbf{F}_p)$ is non-trivial. Hence R and $THH(R|\mathbf{F}_p)$ are not equivalent, and thus, R is not thh-étale.

To see that R is étale we need to give another description for R that requires the use of generalized Steenrod operations for E_∞-algebras (see [11] for a reference on Steenrod operations in our context). In fact, we will only need the operation P^0. Recall that it preserves degree and performs the pth power operation on elements in degree 0. By Section 6 of [9], R can be described as the E_∞-algebra free on two generators x (in degree $-n$) and y with $\mathrm{d}x = 0$ and $\mathrm{d}y = x - P^0 x$. Then noting that $P^0 x$ is of the form $e \otimes x^{\otimes p}$, where e is in $E(p)$ (E being the E_∞ operad), we observe that the R-module representing $TAQ(R)$ is modeled by the free R- module on two generators $\bar{x}$ and $\bar{y}$ with

$$\mathrm{d}\bar{x} = 0$$

and

$$\begin{aligned}\mathrm{d}\bar{y} &= \bar{x} - e \otimes [\bar{x} \otimes x \otimes \cdots \otimes x + \cdots x \otimes \cdots \otimes x \otimes \bar{x}] \\ &= \bar{x} - e(1 + a + \cdots + a^{p-1}) \otimes [\bar{x} \otimes x \otimes \cdots \otimes x],\end{aligned}$$

where a is a generator of the cyclic group of p elements. Observe that we have an R-module contraction s given by

$$s(\bar{y}) = 0$$

and

$$s(\bar{x}) = \bar{y} + f \otimes [\bar{x} \otimes x \otimes \cdots \otimes x],$$

where f is such that $df = e(1 + a + \cdots + a^{p-1})$. Thus $TAQ(R)$ is contractible.

In the following lemma we prove a few easy properties of étale maps that will be needed later.

Lemma 3.6. 1. (*Transitivity*) *If A is étale over R and B is étale over A, then B is étale over R.*

2. (*Base Change*) *If B and C are cofibrant A algebras and B is étale over A, then $C \wedge_A B$ is étale over C. Also, if $B \to C$ is a étale map of A-algebras and D is a cofibrant A-algebra then $B \wedge_A D \to C \wedge_A D$ is also étale.*

3. (*Polynomial Extensions*) *If $B \to C$ is a étale map of A-algebras, then for all cell A-modules X, the induced map $\mathbb{P}_B(X \wedge_A B) \to \mathbb{P}_C(X \wedge_A C)$ is also étale.*

While the lemma and the following proof are stated for étale extensions, a similar result holds for thh-étale algebras as well. The remark after the lemma describes how to adjust the proof for the thh-étale case.

Proof 1. The transitivity is immediate from the cofibration sequence induced by $R \to A \to B$:

$$TAQ(A|R) \wedge_A B \longrightarrow TAQ(B|R) \longrightarrow TAQ(B|A).$$

2. By Proposition 4.6 of [1], $TAQ(C \wedge_A B|C) \simeq TAQ(B|A) \wedge_A C$. Since $TAQ(B|A) \simeq *$, $TAQ(C \wedge_A B|C)$ is also contractible. Now let $B \to C$ be an étale map, then for any A-algebra D,

$$TAQ(C \wedge_A D|B \wedge_A D) \simeq TAQ(C \wedge_B B \wedge_A D|B \wedge_A D) \simeq TAQ(C|B) \wedge_B B \wedge_A D.$$

Here the second map is an equivalence by Proposition 4.6 of [1] once again. Recalling that the map $C \to B$ is étale, we conclude that $TAQ(C \wedge_A D|B \wedge_A D) \simeq *$.

3. It is immediate from Part 2, once we observe that $\mathbb{P}_B(X \wedge_A B) \cong \mathbb{P}_A(X) \wedge_A B$. $\square$

Remark 3.7. Note that the proof of Lemma 3.6 (étale case) hinges on two key facts about TAQ:

1. For cofibrant A algebras B and C, $TAQ(C \wedge_A B|C) \simeq TAQ(B|A) \wedge_A C$.
2. If the map of A-algebras $C \to D$ is étale then $TAQ(C|A) \wedge_C D \simeq TAQ(D|A)$.

Thus, if analogous results hold for THH, then the arguments of the above proof can be repeated to prove the lemma in the thh-étale case. In fact, this reasoning also extends to future results (e.g. Lemma 4.2), in which the étale assumption may be replaced by the thh-étale one.

To see the analogue of the first fact about THH, simply recall the definition of THH that mimics the standard complex for the computation of algebraic Hochschild homology (see [5]). Then $THH(C \wedge_A B|C)$ and $THH(B|A) \wedge_A C$ both have $B \wedge_A \cdots \wedge_A B \wedge_A C$ as simplices and the map between them is the identity map on simplicial level.

Thus the two objects are equivalent. The analogue of the second fact (with some extra conditions) is listed as Lemmas 5.5 and 5.7 and will be proved later.

We have the following result about étale maps.

Proposition 3.8. 1. *If $A \to B$ is an étale map of discrete algebras, then the induced map of S-algebras $HA \to HB$ is also étale.*

2. *If for a commutative ring $k, h: Hk \to B$ is an étale map of S-algebras, then the map $Hk \to H\pi_0(B)$ which realizes the map induced by h on π_0 is also étale.*

Proof 1. Let $A \to B$ is an étale map of discrete algebras. We need to show that $TAQ(HB|HA)$ is contractible. Since HA and HB are connective this is equivalent to showing that the natural map $\phi: HB \to THH(HB|HA)$ is a weak equivalence.

Since $A \to B$ is étale, it is in particular flat, hence by Theorem IX.1.7 of [5], $\pi_*(THH(HB|HA)) \cong HH_*(B|A)$. However for étale maps we have that $HH_0(B|A) \cong B$ and $HH_*(B|A) \cong \Omega^*_{B|A} = 0$ for $* > 0$. Thus, ϕ induces an isomorphism on π_* for $* > 0$ as it is simply the unique map between trivial groups. Combining this with the fact that ϕ on π_0 is the identity map on B, we conclude that ϕ is a weak equivalence.

2. Let $Hk \to B$ be an étale map of S-algebras. B is a generalized Eilenberg–MacLane spectrum since it is a module over the Eilenberg–MacLane spectrum Hk. Hence there is a map $f: H\pi_0(B) \to B$ that realizes the identity map on π_0. Also, for the same reason, we have a map of Hk-algebras $g: B \to H\pi_0(B)$ that induces the identity map on $\pi_0(B)$. The sequence $H\pi_0(B) \to B \to H\pi_0(B)$ produces a pair of maps:

$$TAQ(H\pi_0(B)|Hk) \longrightarrow TAQ(B|Hk) \longrightarrow TAQ(H\pi_0(B)|Hk). \tag{5}$$

Since $g \circ f$ is the identity, the composite map (5) is also an equivalence. However, $TAQ(B|Hk) \simeq *$, since $Hk \to B$ is étale. Hence $TAQ(H\pi_0(B)|Hk) \simeq *$, proving that $Hk \to H\pi_0(B)$ is étale. □

We already mentioned that localizations provide a large class of examples of (thh-)étale maps. As in discrete algebra, another principal source of examples is given by Galois extensions. The following definition is due to John Rognes [14].

Definition 3.9. Let B be a cofibrant A-algebra, and G be a grouplike topological monoid acting on B through A-algebra maps, such that $G \simeq \pi_0(G)$ is finite. Then $A \to B$ is a G-Galois extension if

(1) $A \simeq B^{hG} = F(EG_+, B)^G$, and
(2) $B \wedge_A B \simeq F(G_+, B)$,

where F is the internal function spectrum (see Section I.7 of [5]).

Proposition 3.10 (Rognes [14]). *A G-Galois extension $A \to B$ is thh-étale (and hence also étale).*

Proof. $B \wedge_A B \simeq F(G_+, B)$ is a product of copies of B so B is a retract of $B \wedge_A B$. Hence the composite $B \to THH(B|A) = THH(B, B|A) \to THH(B, B \wedge_A B|A) \simeq B$ is an equivalence and the last map splits (via the retract map). Moreover, since $B \wedge_A B$ is a product of copies of B that map is also a monomorphism in the derived category. Hence, $B \to THH(B|A)$ is an equivalence. □

For examples of Galois extension we again refer to [14].

4. Smooth S-algebras

Definition 4.1. The map of algebras $f : R \to A$ is (thh-)smooth if there is a (thh-)étale covering $\{A \to A_\alpha\}_{\alpha \in \mathscr{I}}$ of A such that for each α there is a factorization

$$R \longrightarrow \mathbb{P}_R X \xrightarrow{\phi} A_\alpha,$$

where X is a cell R-module and $\mathbb{P}_R X$ is the free commutative R-algebra generated by X, with ϕ (thh-)étale.

As always, we would like the smooth S-algebras to generalize the corresponding notion from discrete algebra. Let $k \to A$ be a smooth map of discrete algebras, in other words, for each prime ideal of A, there is an element f away from it such that there is a factorization $k \to k[x_1, \ldots, x_n] \xrightarrow{\phi} A_f$ with ϕ étale. We claim that $Hk \to HA$ is a smooth map of S-algebras. Indeed, we have a pair of maps $Hk \to Hk[x_1, \ldots, x_n] \xrightarrow{H\phi} HA_f$, where $H\phi$ is étale by Proposition 3.8. By the same proposition, we also get that $HA \to HA_f$ is étale. Moreover, the maps $HA \to HA_f$ form a covering, as smashing with HA_f over HA is equivalent to localizing at f. Thus, observing that $Hk[x_1, \ldots, x_n] \cong \mathbb{P}_{Hk}(\bigvee_n Hk)$, we conclude that $Hk \to HA$ is smooth.

In the following lemma we list some of the basic properties of (thh-)smooth S-algebras. Before doing so, we recall that the localization at a cell R-module E is called smashing if for all cell R-modules M, the localization of M at E is given by $R_E \wedge_R M$, where R_E is the localization of R at E.

Lemma 4.2. 1. (*Localization*) *If A is* (*thh-*)*smooth over R and the localization at E is smashing*, *then the composite map $R \to A_E$ is also* (*thh-*)*smooth*.

2. (*Transitivity*) *If A is* (*thh-*)*smooth over R and B is* (*thh-*)*smooth over A*, *then B is* (*thh-*)*smooth over R*.

3. (*Base Change*) *If A is* (*thh-*)*smooth over R*, *and $R \to B$ is a map of commutative S-algebras*, *then $B \to A \wedge_R B$ is also* (*thh-*)*smooth*.

Proof. Again, we present a proof of the smooth case. As noted in Remark 3.7, the proof of thh-smooth case is identical to this one.

Since the localization at E is smashing, $A_E \wedge_A A_E$ is the localization of A_E at E. However, A_E is already E-local. Hence the multiplication map $A_E \wedge_A A_E \to A_E$ is an equivalence, implying that $TAQ(A_E|A) \simeq *$. In other words, $A \to A_E$ is étale. Thus, it is smooth, since for the étale covering required by the definition of smoothness we can simply take the identity map of A_E. So the localization property will follow once we prove the transitivity of smooth algebras.

2. Let $A \to A_\alpha$ and $B \to B_\beta$ be étale coverings of A and B respectively such that there are factorizations $R \to \mathbb{P}_R(X_\alpha) \xrightarrow{\phi_\alpha} A_\alpha$ and $A \to \mathbb{P}_A(Y_\beta) \xrightarrow{\psi_\beta} B_\beta$ with ϕ_α and ψ_β étale. Consider the maps

$$B \to B_\beta \wedge_A A_\alpha \wedge_{\mathbb{P}_R(X_\alpha)} A_\alpha. \tag{6}$$

By Part 2 of Lemma 3.6, we have that the maps $B \to B_\beta \wedge_A A_\alpha$ and $A_\alpha \to A_\alpha \wedge_{\mathbb{P}_R(X_\alpha)} A_\alpha$ are étale. Hence, the map $B_\beta \wedge_A A_\alpha \to B_\beta \wedge_A A_\alpha \wedge_{\mathbb{P}_R(X_\alpha)} A_\alpha$ also étale. Thus, by transitivity of étale extensions (Part 1 of Lemma 3.6), we get that the above maps 6 are étale. Next we show that this collection of étale maps forms a covering. To see this first observe that since $A \to A_\alpha$ is a covering of A, so is $A \to A_\alpha \wedge_{\mathbb{P}_R(X_\alpha)} A_\alpha$, as the multiplication map $A_\alpha \wedge_{\mathbb{P}_R(X_\alpha)} A_\alpha \to A_\alpha$ splits. Now let $M \to N$ be B-modules such that $M \wedge_B B_\beta \wedge_A A_\alpha \wedge_{\mathbb{P}_R(X_\alpha)} A_\alpha \xrightarrow{\cong} N \wedge_B B_\beta \wedge_A A_\alpha \wedge_{\mathbb{P}_R(X_\alpha)} A_\alpha$. Since $A \to A_\alpha \wedge_{\mathbb{P}_R(X_\alpha)} A_\alpha$ is a covering, we conclude that for each β, $M \wedge_B B_\beta \xrightarrow{\cong} N \wedge_B B_\beta$, and hence $M \simeq N$.

Thus, it remains to show that $B_\beta \wedge_A A_\alpha \wedge_{\mathbb{P}_R(X_\alpha)} A_\alpha$ is étale over a polynomial extension of R. By Part 2 of Lemma 3.6 we have that $\mathbb{P}_A(Y_\beta) \wedge_A A_\alpha \wedge_{\mathbb{P}_R(X_\alpha)} A_\alpha \to B_\beta \wedge_A A_\alpha \wedge_{\mathbb{P}_R(X_\alpha)} A_\alpha$ is étale. Now we simply observe that $\mathbb{P}_A(Y_\beta) \wedge_A A_\alpha \wedge_{\mathbb{P}_R(X_\alpha)} A_\alpha \cong \mathbb{P}_{A_\alpha}(Y_\beta \wedge_A A_\alpha) \wedge_{\mathbb{P}_R(X_\alpha)} A_\alpha \cong \mathbb{P}_{\mathbb{P}_R(X_\alpha)}(Y_\beta \wedge_A A_\alpha \wedge_{\mathbb{P}_R(X_\alpha)} A_\alpha)$ and the last object being a polynomial extension of a polynomial over R is itself a polynomial over R.

3. Let $A \to A_\alpha$ be an étale covering of A such that there are factorizations $R \to \mathbb{P}_R(X_\alpha) \xrightarrow{\phi_\alpha} A_\alpha$ with ϕ_α étale. For any R-algebra B, by Part 2 of Lemma 3.6, the maps $B \wedge_R A \to B \wedge_R A_\alpha$ are étale. Moreover, since $\mathbb{P}_R(X_\alpha) \xrightarrow{\phi_\alpha} A_\alpha$ are étale, so are $\mathbb{P}_R(X_\alpha) \wedge_R B \to A_\alpha \wedge_R B$. Note that $\mathbb{P}_R(X_\alpha) \wedge_R B \cong \mathbb{P}_B(X_\alpha \wedge_R B)$. Thus, we have factorizations $B \to \mathbb{P}_B(X_\alpha \wedge_R B) \xrightarrow{\psi_\alpha} A_\alpha \wedge_R B$ with ψ étale.

To complete the proof it remains to show that the collection of étale maps $B \wedge_R A \to B \wedge_R A_\alpha$ forms an étale covering. Let $M \to N$ be a pair of $B \wedge_R A$-modules such that $M \wedge_{B \wedge_R A} B \wedge_R A_\alpha \xrightarrow{\simeq} N \wedge_{B \wedge_R A} B \wedge_R A_\alpha$. Observe that

$$M \wedge_{B \wedge_R A} B \wedge_R A_\alpha \cong M \wedge_{B \wedge_R A} B \wedge_R A \wedge_A A_\alpha \cong M \wedge_A A_\alpha.$$

Thus we get that $M \wedge_A A_\alpha \xrightarrow{\simeq} N \wedge_A A_\alpha$, and since $A \to A_\alpha$ is an étale covering, we conclude that $M \simeq N$. □

5. Étale descent

Our main goal is to prove the topological analogue of the HKR theorem. As will be observed later, it is of critical importance for HKR that we be able to identify conditions on the map of R-algebras $A \to B$ that will imply the identity $THH(A|R) \wedge_A B \simeq THH(B|R)$. In fact recalling that by McClure et al. [12] $THH(A|R) \cong A \otimes_R S^1$, we can rewrite the above identity as $(A \otimes_R S^1) \wedge_A B \simeq B \otimes_R S^1$, which prompts us to investigate conditions on a simplicial set X and a map of R-algebras $A \to B$ that imply the more general identity

$$(A \otimes_R X) \wedge_A B \simeq B \otimes_R X. \tag{7}$$

Almost immediately we can get a necessary condition for (7) to hold. First we need a change of base formula for tensor products

$$A \wedge_{A \otimes_R X} (B \otimes_R X) \simeq B \otimes_A X. \tag{8}$$

We are grateful to M. Mandell for suggesting a proof of this formula by describing the A-algebra maps into a fixed A-algebra C.

First consider $\mathscr{C}_A(A \wedge_{A \otimes_R X} (B \otimes_R X), C)$. By universal property of pushouts, this is isomorphic to the subset of maps f in $\mathscr{C}_R(B \otimes_R X, C)$, such that the restriction of f to $A \otimes_R X$ factors through $A \otimes_R X \to A$. By adjunction of the tensor product, $\mathscr{C}_R(B \otimes_R X, C) \cong \mathscr{U}(X, \mathscr{C}_R(B, C))$. Thus, $\mathscr{C}_A(A \wedge_{A \otimes_R X} (B \otimes_R X), C)$ is isomorphic to the subset of maps ϕ in $\mathscr{U}(X, \mathscr{C}_R(B, C))$ such that for all $x \in X, \phi(x) : B \to C$ restricted to A is the same map, in other words, the maps $A \to B \overset{\phi(x)}{\to} C$ and $A \to B \overset{\phi(y)}{\to} C$ are the same for all $x, y \in X$. Observe that the collection of such maps is precisely $\mathscr{U}(X, \mathscr{C}_A(B, C)) \cong \mathscr{C}_A(B \otimes_A X, C)$, and hence the proof of formula (8) is complete by Yoneda's lemma.

Now consider the following commutative diagram of A-algebras

$$\begin{tikzcd}
A \wedge_{A \otimes_R X} (A \otimes_R X) \wedge_A B \arrow[r] \arrow[d] & A \wedge_{A \otimes_R X} (B \otimes_R X) \arrow[d] \\
B \arrow[r] & B \otimes_A X.
\end{tikzcd}$$

The left vertical arrow is clearly an isomorphism, and by the base change formula (8), so is the right vertical arrow. Hence if we assume that identity (7) holds, then the top horizontal map is an equivalence, implying that the bottom map $B \to B \otimes_A X$ is also an equivalence.

Thus, $B \overset{\simeq}{\to} B \otimes_A X$ is a necessary condition for (7) to hold. Of course, in general this condition alone is not enough to ensure (7), as can easily be seen an example of $X = S^0$. $B \otimes_A S^0 \cong B \wedge_A B$, and hence the condition $B \overset{\simeq}{\to} B \otimes_A X$ becomes $B \overset{\simeq}{\to} B \wedge_A B$. This, however, as can be seen in the following example, does not imply $A \wedge_R B \simeq B \wedge_R B$, which is the restatement of (7) for $X = S^0$.

Example 5.1. Consider a pair of S-algebra maps $H\mathbf{Z} \to H\mathbf{Z}/p\mathbf{Z} \to H\mathbf{Z}_\mathbf{p}^\wedge$, where $\mathbf{Z}$ is the integers and $\mathbf{Z}_\mathbf{p}^\wedge$ is the p-completion of $\mathbf{Z}$, i.e. the ring of p-adic numbers. In other words, in the above setup, we have taken R, A and B to be $H\mathbf{Z}$, $H\mathbf{Z}/p\mathbf{Z}$ and $H\mathbf{Z}_\mathbf{p}^\wedge$, respectively. First observe that $B \wedge_A B \simeq B$. Indeed, by Theorem IV.2.1 of [5], we have that

$$\pi_*(H\mathbf{Z}_\mathbf{p}^\wedge \wedge_{H\mathbf{Z}/p\mathbf{Z}} H\mathbf{Z}_\mathbf{p}^\wedge) \cong Tor_*^{\mathbf{Z}/p\mathbf{Z}}(\mathbf{Z}_\mathbf{p}^\wedge, \mathbf{Z}_\mathbf{p}^\wedge).$$

Hence, $\pi_0(B \wedge_A B) = \pi_0(H\mathbf{Z}_\mathbf{p}^\wedge \wedge_{H\mathbf{Z}/p\mathbf{Z}} H\mathbf{Z}_\mathbf{p}^\wedge) \cong \mathbf{Z}_\mathbf{p}^\wedge \otimes_{\mathbf{Z}/p\mathbf{Z}} \mathbf{Z}_\mathbf{p}^\wedge$. However, by Theorem 7.2 of [4], $\mathbf{Z}_\mathbf{p}^\wedge \otimes_{\mathbf{Z}} /p\mathbf{Z}\mathbf{Z}_\mathbf{p}^\wedge$ is isomorphic to the p-adic completion of $\mathbf{Z}_\mathbf{p}^\wedge$, and since $\mathbf{Z}_\mathbf{p}^\wedge$ is already complete, we conclude that $\pi_0(B \wedge_A B) \cong \mathbf{Z}_\mathbf{p}^\wedge$. As for $\pi_*(B \wedge_A B)$ for $* > 0$, they are all trivial, since by Theorem 7.2 of [4], $\mathbf{Z}_\mathbf{p}^\wedge$ is flat over $\mathbf{Z}/p\mathbf{Z}$, and hence $Tor_*^{\mathbf{Z}}/p\mathbf{Z}(\mathbf{Z}_\mathbf{p}^\wedge, \mathbf{Z}_\mathbf{p}^\wedge) \cong 0$ for $* > 0$. Thus, we conclude that $B \wedge_A B \simeq B$.

To prove that $A \wedge_R B$ and $B \wedge_R B$ are not weakly equivalent, it is enough to show that $\pi_0(A \wedge_R B)$ is not isomorphic to $\pi_0(B \wedge_R B)$, which is evident, since

$$\pi_0(A \wedge_R B) \cong (\mathbf{Z}/p\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Z}_{\mathbf{p}}^{\wedge} \cong \mathbf{Z}_{\mathbf{p}}^{\wedge}/p\mathbf{Z}_{\mathbf{p}}^{\wedge} \cong \mathbf{Z}/p\mathbf{Z},$$

while $\pi_0(B \wedge_R B) \cong \mathbf{Z}_{\mathbf{p}}^{\wedge} \otimes_{\mathbf{Z}} \mathbf{Z}_{\mathbf{p}}^{\wedge}$.

To produce a sufficient condition for (7) to hold, first we set up the notation, then introduce a few key identities which, if true, would imply Eq. (7). We discuss conditions under which these identities hold and, to conclude the section, summarize our findings in two (étale descent) lemmas.

The objective is to compare the algebras $B \otimes_R X$ and $(A \otimes_R X) \wedge_A B$. We do this by comparing two towers of objects that approximate $B \otimes_R X$ and $(A \otimes_R X) \wedge_A B$, respectively. For the special case $X = S^1$ such towers were considered in [7] for the category of chain complexes and adopted to the category of S-algebras in [13].

Fix a simplicial set X. Define I_A to be the hofiber of the multiplication map $A \otimes_R X \to A$. Then I_A inherits a multiplicative structure and we define I_A/I_A^n by the pushout diagram

$$\begin{array}{ccc} I_A^n & \longrightarrow & I_A \\ \downarrow & & \downarrow \\ * & \longrightarrow & I_A/I_A^n, \end{array}$$

where the smash powers of I_A are taken over $A \otimes_R X$.

Proposition 5.2. *Let $A \to B$ be thh-étale. Then the towers $\{(I_A/I_A^n) \wedge_A B\}$ and $\{I_B/I_B^n\}$ are weakly equivalent. Consequently,*

$$holim[(I_A/I_A^n) \wedge_A B] \simeq holim\, I_B/I_B^n.$$

Proof. We begin by showing that

$$I_B/I_B^2 \simeq I_A/I_A^2 \wedge_A B. \tag{9}$$

To see this, we employ new notation to denote the fiber of $A \otimes_R X \to A$ by I_X whenever we wish to consider it as a functor of simplicial sets, as opposed to R-algebras. Observe that since I_X/I_X^2 is a linear functor and $X \cong S^0 \wedge X$, we have that I_X/I_X^2 is equivalent to $I_{S^0}/I_{S^0}^2 \wedge X$. Recall that $I_{S^0}/I_{S^0}^2 \simeq \Sigma TAQ(A|R)$ (see e.g. [13]). Thus, to show (9), it suffices to prove that

$$\Sigma TAQ(B|R) \wedge X \simeq \Sigma TAQ(A|R) \wedge_A B \wedge X,$$

which, in turn, is an immediate consequence of the transfer sequence of TAQ:

$$TAQ(A|R) \wedge_A B \longrightarrow TAQ(B|R) \longrightarrow TAQ(B|A),$$

combined with the fact that $TAQ(B/A) \simeq *$ since $A \to B$ is thh-étale.

To complete the proof, we induct on n. Suppose the natural map $I_A/I_A^{n-1} \wedge_A B \to I_B/I_B^{n-1}$ is a weak equivalence. By naturality, we have a commutative diagram

$$\begin{array}{ccccc} (I_A^{n-1}/I_A^n) \wedge_A B & \longrightarrow & (I_A/I_A^n) \wedge_A B & \longrightarrow & (I_A/I_A^{n-1}) \wedge_A B, \\ \downarrow & & \downarrow & & \downarrow \\ I_B^{n-1}/I_B^n & \longrightarrow & I_B/I_B^n & \longrightarrow & I_B/I_B^{n-1} \end{array}$$

where the objects in the left column are the hofibers of the right maps. Since both rows are (co)fibration sequences and the right vertical map is a weak equivalence by inductive assumption, it is enough to show that the left vertical map is also a weak equivalence. This, however, is an immediate consequence of Proposition 2.4 of [13], which states that

$$\left[\bigwedge_A^n I_A/I_A^2\right]_{h\Sigma_n} \simeq I_A^n/I_A^{n+1},$$

where the lower script A in the above smash product on the left indicates that the smash product is taken over A. Thus, we have a series equivalences

$$(I_A^n/I_A^{n+1}) \wedge_A B \simeq \left[\bigwedge_A^n I_A/I_A^2\right]_{h\Sigma_n} \wedge_A B \simeq \left[\bigwedge_B^n I_A/I_A^2 \wedge_A B\right]_{h\Sigma_n} \simeq \left[\bigwedge_B^n I_B/I_B^2\right]_{h\Sigma_n}$$
$$\simeq I_B^n/I_B^{n+1},$$

which proves that the left vertical arrow, and consequently the middle one, are weak equivalences. □

Observe that, in view of the above proposition, the étale descent formula (7) will hold if $A \otimes_R X \wedge_A B$ and $B \otimes_R X$ are equivalent to $holim[(I_A/I_A^n) \wedge_A B]$ and $holim[(I_B/I_B^n)$, respectively. To address this, we pause to discuss completions and complete objects in our framework.

Definition 5.3. 1. Let A be a cofibrant R-algebra. Define the completion $(A \otimes_R X)^\wedge$ of $A \otimes_R X$ to be the inverse limit $holim(A \otimes_R X)/I_A^n$.

2. For an $A \otimes_R X$-module M, the completion $M^\wedge$ of M is defined to be $holim(M/I_A^n)$, where M/I_A^n is the cofiber of the obvious map $I_A^n \wedge_{A\otimes_R X} M \to (A \otimes_R X) \wedge_{A\otimes_R X} M \xrightarrow{\cong} M$. Here as before the powers of I_A are taken over $A \otimes_R X$.

3. M is complete if the natural map $M \to holim[M/I_A^n]$ is a week equivalence.

The following result helps to transmit information between an S-algebra and its completion.

Proposition 5.4. *If M is a finite A-CW-complex, then the natural map*

$$(A \otimes_R X)^\wedge \wedge_{A\otimes_R X} M \otimes_R X \longrightarrow (M \otimes_R X)^\wedge$$

is an equivalence.

Consequently, if B is a thh-étale algebra over A, which is a finite A-CW-complex when viewed as an A-module, then the completion of $B \otimes_R X$ with respect to I_B is weakly equivalent to the completion of $B \otimes_R X$ viewed as an $A \otimes_R X$-module.

Proof. The proposition is clearly true for $M = A$. Observe that as a consequence of adjunctions

$$\mathscr{C}_R(A \otimes_R X, B) \cong \mathscr{U}(X, \mathscr{C}_R(A, B)) \cong \mathscr{C}_R(A, F(X_+, B)),$$

we have that $(\Sigma^i A) \otimes_R X \cong \Sigma^i(A \otimes_R X)$, where $\Sigma^i A$ is the i'th suspension of A; and in the above adjunctions $\mathscr{C}_R$ and $\mathscr{U}$ are the categories of commutative R-algebras and unbased spaces, respectively. Hence, $((\Sigma^i A) \otimes_R X)^\wedge \simeq \Sigma^i(A \otimes_R X)^\wedge$, i.e. the proposition holds for suspensions of A as well.

Now suppose, the statement is true for some module K and let F be a wedge of sphere modules S^i_A with a hocofiber N:

$$F \longrightarrow K \longrightarrow N. \tag{10}$$

Consider the following commutative diagram

$$\begin{tikzcd}
(A \otimes_R X)^\wedge \wedge_{A\otimes_R X} F \arrow[r] \arrow[d] & (A \otimes_R X)^\wedge \wedge_{A\otimes_R X} K \arrow[r] \arrow[d] & (A \otimes_R X)^\wedge \wedge_{A\otimes_R X} N \arrow[d] \\
F^\wedge \arrow[r] & K^\wedge \arrow[r] & N^\wedge
\end{tikzcd}$$

Note that both rows are cofibrations and the two left vertical maps are weak equivalences—the first one by our above discussion on suspensions of A, and the second one by assumption on K. Hence the right vertical map $(A \otimes_R X)^\wedge \wedge_{A\otimes_R X} N \to N^\wedge$ is also a weak equivalence, which proves the first part of the proposition, as A-CW-complexes are built precisely via sequences (10).

To prove the second part of the proposition, we apply $- \wedge_{A\otimes_R X} (B \otimes_R X)$ to the sequence $I_A \to A \otimes_R X \to A$ to get a cofibration sequence

$$I_A \wedge_{A\otimes_R X} (B \otimes_R X) \to (A \otimes_R X) \wedge_{A\otimes_R X} (B \otimes_R X) \to A \wedge_{A\otimes_R X} (B \otimes_R X).$$

Note that by the base change formula (8) for tensor products, the last term $A \wedge_{A\otimes_R X} B \otimes_R X$ is equivalent to $B \otimes_A X$, which, in turn, is weakly equivalent to B by thh-étale assumption. Hence we have a cofibration sequence

$$I_A \wedge_{A\otimes_R X} (B \otimes_R X) \longrightarrow B \otimes_R X \longrightarrow B$$

and are, thus, entitled to conclude that $I_A \wedge_{A\otimes_R X} (B \otimes_R X) \simeq I_B$. The conclusion follows from the first part of the proposition. □

We are ready to state our first étale descent lemma.

Lemma 5.5 (étale descent, complete case). *Let A be a cofibrant R-algebra, such that $A \otimes_R X$ is complete, and B be a cofibrant A-algebra which is a finite A-CW-complex when viewed as an A-module. Then $A \to B$ is thh-étale if and only if the etale descent formula holds*:

$$(A \otimes_R X) \wedge_A B \simeq B \otimes_R X.$$

Proof. We only need to prove the 'only if' direction. Let $R \to A \to B$ be as in the lemma, with $A \to B$ thh-étale. Then, by definition of completeness and due to the fact that smashing with finite CW-complexes commutes with holims, we have

$$(A \otimes_R X) \wedge_A B \simeq holim[(A \otimes_R X)/I_A^n] \wedge_A B \simeq holim[((A \otimes_R X)/I_A^n) \wedge_A B]. \tag{11}$$

Recall that by Proposition 5.2,

$$holim[((A \otimes_R X)/I_A^n) \wedge_A B] \simeq holim[(B \otimes_R X)/I_B^n]. \tag{12}$$

Hence, it remains to prove that $holim[(B \otimes_R X)/I_B^n]$ is weakly equivalent to $B \otimes_R X$, or in other words, that $B \otimes_R X$ is complete with respect to I_B, which, of course, is equivalent to being complete as an $A \otimes_R X$-module by Proposition 5.4. Denote the homotopy fiber of the natural map $A \otimes_R X \to (A \otimes_R X)^\wedge$ by K and consider the following diagram whose right column is obtained by applying $- \wedge_{A\otimes_R X} (B \otimes_R X)$ to the cofiber sequence $K \to A \otimes_R X \to (A \otimes_R X)^\wedge$:

$$\begin{tikzcd}
K \wedge_A B \arrow[r] \arrow[d] & K \wedge_{A\otimes_R X} (B \otimes_R X) \arrow[d] \\
(A \otimes_R X) \wedge_A B \arrow[r] \arrow[d] & (A \otimes_R X) \wedge_{A\otimes_R X} (B \otimes_R X) \cong B \otimes_R X \arrow[d] \\
(A \otimes_R X)^\wedge \wedge_A B \arrow[r] & (A \otimes_R X)^\wedge \wedge_{A\otimes_R X} (B \otimes_R X) \simeq (B \otimes_R X)^\wedge.
\end{tikzcd}$$

Since $A \otimes_R X$ is complete, K is contractible; hence the top row is a weak equivalence. The bottom row is also an equivalence since combining Eqs. (11) and (12) we get

$$(A \otimes_R X)^\wedge \wedge_A B \simeq (A \otimes_R X) \wedge_A B \simeq holim[(B \otimes_R X)/I_B^n] \simeq (B \otimes_R X)^\wedge.$$

Hence, we are allowed to conclude that the middle row is also an equivalence, which proves the lemma. □

Remark 5.6. We would like to point out that it is this étale descent lemma that prompted us to consider the thh-étale algebras (in addition to étale ones). Of course, the more direct translation of the 'étale' notion from discrete algebra appears to be what we have defined as étale S-algebras, since in both cases étale essentially means unramified, i.e. with a vanishing module of differentials. Hence, perhaps one would like/hope to prove an étale descent lemma with an étale condition (as opposed to a slightly stronger thh-étale requirement as we have imposed). However, as we have demonstrated, the (stronger) thh-étale condition is a necessary one. We also note that the notion of thh-étale maps is also a generalization of étale maps from discrete algebra; in fact, as pointed out earlier, when restricted to Eilenberg–MacLane spectra étale and thh-étale coincide.

We return to the completeness assumption in the étale descent lemma above. That assumption is satisfied if A is connective and the simplicial set X is such that $\pi_0(X)=0$, as clearly the connectivity of maps

$$A \otimes_R X \longrightarrow (A \otimes_R X)/I_A^n$$

increases with n, since with A connective and X connected, I_A is at least 1-connected. Equivalently, the connectivity of fibers I_A^n/I_A^{n+1} increases with n. Moreover, if B is a connective A-algebra then by Eilenberg–Moore spectral sequence (Section 4, Chapter IV of [5]), the connectivity of the maps

$$A \otimes_R X \wedge_A B \longrightarrow ((A \otimes_R X)/I_A^n) \wedge_A B$$

also increases with n, which implies that

$$A \otimes_R X \wedge_A B \simeq holim[((A \otimes_R X)/I_A^n) \wedge_A B].$$

By Proposition 5.4, $holim[((A \otimes_R X)/I_A^n) \wedge_A B]$ is weakly equivalent to $holim[(B \otimes_R X)/I_B^n]$, which, in turn is equivalent to $B \otimes_R X$ since B is connective and X is connected, and hence, $B \otimes_R X$ is complete.

We have proved the following lemma.

Lemma 5.7 (étale descent, connective case). *Let A be a connective cofibrant R-algebra, B a connective cofibrant A-algebra, and X a connected simplicial set. Then $A \to B$ is thh-étale if and only if the étale descent formula folds*:

$$(A \otimes_R X) \wedge_A B \simeq B \otimes_R X.$$

In conclusion of this section, we present a result that helps to detect the condition $B \otimes_A X \simeq B$ necessary (and often sufficient) for the étale descent Lemmas 5.5 and 5.7 to hold. We set up the notation first.

For a simplicial set X_*, let J_X be the fiber of the obvious (induced by multiplication) map $B \otimes_A X \to B$ to emphasize that J is a functor of simplicial sets.

Proposition 5.8. *Let $A \to B$ be a map of commutative R-algebras and X_* a simplicial set such that $B \otimes_A X$ is complete with respect to J_X. Then $B \otimes_A X \simeq B$ if and only if $H_*(X, TAQ(B|A)) = 0$ for all $*$.*

Proof. We begin by observing that $B \otimes_A X \simeq B$ if and only if $J_X \simeq *$. This in turn implies that $J_X/J_X^2 \simeq *$. Furthermore, the converse of this is also true. Indeed, let $J_X/J_X^2 \simeq *$. By Johnson and McCarthy [7] or Minasian [13] we have that

$$hofiber(J_X/J_X^{n+1} \longrightarrow J_X/J_X^n) \simeq [(J_X/J_X^2)^{\wedge n}]_{h\Sigma_n}.$$

This result is listed as Proposition 2.4 in [13], which in turn is the adaptation to the framework of S-algebras of a similar result obtained in [7] for the category of chain complexes. Now if $J_X/J_X^2 \simeq *$ then the first term and all homotopy fibers in the inverse limit system $\{J_X/J_X^n\}$ are contractible. Hence, $J_X \simeq holim\, J_X/J_X^n \simeq *$.

Recalling that the term J_X/J_X^2 is linear and that $X \cong X \wedge S^0$, we get an identity $J_X/J_X^2 \simeq X \wedge J_{S^0}/J_{S^0}^2$. Thus, $B \otimes_A X \simeq B$ if and only if $X \wedge J_{S^0}/J_{S^0}^2 \cong *$. To complete the proof it remains to observe that $TAQ(B|A) \simeq J_{S^0}/J_{S^0}^2$, and hence $B \otimes_A X \simeq B$ is equivalent to $X \wedge TAQ(B|A) \simeq *$, or in other words, to $H_*(X, TAQ(B|A)) = 0$ for all $*$. □

6. HKR theorem

Theorem 6.1. *Let $f : R \to A$ be thh-smooth in the category of connective S-algebras. Then the natural (derivative) map $THH(A|R) \to \Sigma TAQ(A|R)$ has a section in the category of A-modules which induces an equivalent of A-algebras*:

$$\mathbb{P}_A \Sigma TAQ(A|R) \xrightarrow{\simeq} THH(A|R).$$

Proof. First we show that the Theorem holds for polynomial extensions $R \to \mathbb{P}_R X$, where X is a cell R-module. Our first objective is to compute $TAQ(\mathbb{P}_R X|R)$. While one can do this directly from definitions, we present a somewhat more concise computation that employs series of adjunctions. By Proposition 3.2 of [1], for every $\mathbb{P}_R X$-module M,

$$h\mathscr{M}_{\mathbb{P}_R X}(TAQ(\mathbb{P}_R X|R), M) \cong h\mathscr{C}_{R/\mathbb{P}_R X}(\mathbb{P}_R X, \mathbb{P}_R X \vee M),$$

where $\mathscr{C}_{R/\mathbb{P}_R X}$ is the category of R-algebras over $\mathbb{P}_R X$, and $h\mathscr{M}$ and $h\mathscr{C}$ indicate the corresponding homotopy categories. Of course, it is immediate that $\mathscr{C}_{R/\mathbb{P}_R X}(\mathbb{P}_R X, \mathbb{P}_R X \vee M) \cong \mathscr{C}_R(\mathbb{P}_R X, M)$. Furthermore, since the free functions $\mathbb{P}_R$ and $\mathbb{P}_R X \wedge_R$ -(with X a cell R-module) are left adjoints which preserve cofibrations and trivial cofibrations, they induce adjunctions on homotopy categories as well (see [3]). Thus, we get

$$h\mathscr{C}_R(\mathbb{P}_R X, M) \cong h\mathscr{M}_R(X, M) \cong hM_{\mathbb{P}_R X}(\mathbb{P}_R X \wedge_R X, M).$$

Hence, by Yoneda's lemma, we have an equivalence of $\mathbb{P}_R X$ modules $TAQ(\mathbb{P}_R X|R) \simeq \mathbb{P}_R X \wedge_R X$.

On the other hand, by a theorem of McClure et al. [12], $THH(\mathbb{P}_R X|R) \cong \mathbb{P}_R X \otimes_R S^1_*$. We have adjunction homeomorphisms

$$\mathscr{C}_R(\mathbb{P}_R X \otimes_R S^1_*, B) \cong \mathscr{U}(S^1_*, \mathscr{C}_R(\mathbb{P}_R X, B)) \cong \mathscr{U}(S^1, \mathscr{M}_R(X, B)) \cong \mathscr{M}_R(X \wedge S^1_+, B)$$
$$\cong \mathscr{C}_R(\mathbb{P}_R(X \wedge S^1_+), B),$$

where $\mathscr{C}_R$ is the category of commutative R-algebras, $\mathscr{U}$ is the category of unbased topological spaces, and B is a commutative R-algebra. Hence, by Yoneda's lemma, $THH(\mathbb{P}_R X|R) \cong \mathbb{P}_R(X \wedge S^1_+)$ as R-algebras. Of course, $\mathbb{P}_R(X \wedge S^1_+)$ (and consequently $THH(\mathbb{P}_R X|R)$) also has a structure of a $\mathbb{P}_R X$-algebra, which is more evident once we observe that $\mathbb{P}_R(X \wedge S^1_+) \cong \mathbb{P}_R(X \vee \Sigma X) \cong \mathbb{P}_R X \wedge_R \mathbb{P}_R(\Sigma X)$. Finally, note that by the base change formula for polynomial algebras, we have

$$\mathbb{P}_R(X \wedge S^1_+) \cong \mathbb{P}_R X \wedge_R \mathbb{P}_R(\Sigma X) \cong \mathbb{P}_{\mathbb{P}_R X}(\mathbb{P}_R X \wedge_R \Sigma X).$$

Hence, recalling that $TAQ(\mathbb{P}_R X|R) \simeq \mathbb{P}_R X \wedge_R X$, we are allowed to conclude that as $\mathbb{P}_R X$-algebras Topological Hochschild Homology $THH(\mathbb{P}_R X|R)$ is equivalent to $\mathbb{P}_{\mathbb{P}_R X}(\mathbb{P}_R X \wedge_R \Sigma X) \cong \mathbb{P}_{\mathbb{P}_R X}(\Sigma TAQ(\mathbb{P}_R X|R))$.

Now let $R \to A$ be an arbitrary smooth map. Thus we have a family of sequences

$$R \longrightarrow \mathbb{P}_R X \xrightarrow{\phi} A_\alpha$$

with ϕ thh-étale. By Basterra [1], this sequences give rise to cofibration sequences

$$TAQ(\mathbb{P}_R X|R) \wedge_{\mathbb{P}_R X} A_\alpha \longrightarrow TAQ(A_\alpha|R) \longrightarrow TAQ(A_\alpha|\mathbb{P}_R X). \tag{13}$$

Since ϕ is thh-étale, the last term of this sequence is 0. Hence,

$$TAQ(\mathbb{P}_R X|R) \wedge_{\mathbb{P}_R X} A_\alpha \xrightarrow{\simeq} TAQ(A_\alpha|R). \tag{14}$$

Similarly, the sequences $R \to A \to A_\alpha$ produce cofibration sequences

$$TAQ(A|R) \wedge_A A_\alpha \to TAQ(A_\alpha|R) \to TAQ(A_\alpha|A). \tag{15}$$

Since the maps $A \to A_\alpha$ are thh-étale by definition, $TAQ(A_\alpha|A)$ are contractible. Hence, we get an equivalence of A-modules

$$TAQ(A|R) \wedge_A A_\alpha \xrightarrow{\simeq} TAQ(A_\alpha|R). \tag{16}$$

Combining the above Lemma 5.7 with the fact that we have proved the theorem for polynomial extensions, we get a series of equivalences

$$THH(A_\alpha|R) \cong THH(\mathbb{P}_R X|R) \wedge_{\mathbb{P}_R X} A_\alpha \cong \mathbb{P}_{\mathbb{P}_R X}(\Sigma TAQ(\mathbb{P}_R X|R)) \wedge_{\mathbb{P}_R X} A_\alpha. \tag{17}$$

Next, observe that $\mathbb{P}_{\mathbb{P}_R X}(\Sigma TAQ(\mathbb{P}_R X|R)) \wedge_{\mathbf{p}_R X} A_\alpha \cong \mathbb{P}_{A_\alpha}(\Sigma TAQ(\mathbb{P}_R X|R) \wedge_{\mathbb{P}_R X} A_\alpha)$, which combined with Eq. (14) gives us the theorem for the extensions $R \to A_\alpha$:

$$THH(A_\alpha|R) \simeq \mathbb{P}_{A_\alpha}(\Sigma TAQ(A_\alpha|R)). \tag{18}$$

To complete the proof, note that Lemma 5.7 applied to the thh-étale map $A \to A_\alpha$ gives an equivalence $THH(A_\alpha|R) \simeq THH(A|R) \wedge_A A_\alpha$; and plugging this and Eq. (16) into the above equivalence (18), we get

$$THH(A|R) \wedge_A A_\alpha \simeq \mathbb{P}_{A_\alpha}(\Sigma TAQ(A|R) \wedge_A A_\alpha) \simeq \mathbb{P}_A(\Sigma TAQ(A|R)) \wedge_A A_\alpha.$$

Recalling the second condition of the definition of thh-étale covers $A \to A_\alpha$, we conclude that $THH(A|R)$ and $\mathbb{P}_A(\Sigma TAQ(A|R))$ are equivalent as A-algebras. □

Theorem 6.2. *Let $B \to R \xrightarrow{f} A$ be maps of connective S-algebras with f thh-smooth. Then the first fundamental sequence of modules of differentials splits, i.e.*

$$TAQ(A|B) \simeq (TAQ(R|B) \wedge_R A) \vee TAQ(A|R).$$

Proof. Consider the suspension of the first fundamental sequence of differential modules for $B \to R \xrightarrow{f} A$:

$$\Sigma TAQ(R|B) \wedge_R A \to \Sigma TAQ(A|B) \to \Sigma TAQ(A|R). \tag{19}$$

By Theorem 6.1, we have a map $\Sigma TAQ(A|R) \to THH(A|R)$ which is a section to the derivative map. The smash product over B of the maps id: $A \to A$ and $B \to R$ induces a map $THH(A|R) \to THH(A \wedge_B R|R) \xrightarrow{\simeq} THH(A|B)$. Thus we get a map

$$\phi : \Sigma TAQ(A|R) \to THH(A|B).$$

Next consider the natural commutative diagram

$$\begin{array}{ccc} THH(A|B) & \longrightarrow & THH(A|R) \\ \big\downarrow & & \big\downarrow \\ \Sigma TAQ(A|B) & \longrightarrow & \Sigma TAQ(A|R). \end{array}$$

It is easy to see that the map $\phi : \Sigma TAQ(A|R) \to THH(A|B)$ is a section to the map from $THH(A|B)$ to $\Sigma TAQ(A|R)$ in the above diagram. Thus, ϕ composed with the derivative map $THH(A|B) \to \Sigma TAQ(A|B)$ gives a map $\Sigma TAQ(A|R) \to \Sigma TAQ(A|B)$ which is a section to the second map in the first fundamental sequence, and thus splits the sequence. This map combined with the first map in the fundamental sequence (19) induces a map

$$(TAQ(R|B) \wedge_R A) \vee TAQ(A|R) \longrightarrow TAQ(A|S). \tag{20}$$

Since the second map in Eq. (19) has a section, it is surjective on homotopy groups and the long exact sequence of homotopy groups associated to the cofibration sequence (19) breaks up into a series of *split* short exact sequences:

$$\pi_i(TAQ(R|B) \wedge_R A) \to \pi_i(TAQ(A|S)) \to \pi_i(TAQ(A|R)).$$

Hence, $\pi_i(TAQ(A|S)) \cong \pi_i(TAQ(R|B)) \wedge_R A) \oplus \pi_i(TAQ(A|R))$, which implies that map (20) induces an isomorphism on homotopy groups and is thus a weak equivalence. □

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